

# Generating Functions for Multi- $j$ -Symbols

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## Abstract

A formula is derived that provides generating functions for any multi- $j$ -symbol, such as the 3- $j$ -symbol, the 6- $j$ -symbol, the 9- $j$ -symbol, etc. The result is completely determined by geometrical objects (loops and curves) in the graph of the multi- $j$ -symbol. A geometric-combinatorial interpretation for multi- $j$ -symbols is given.

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## 1 Introduction

The 3- $j$ -symbol describes how the tensor product of two  $SU(2)$  representations  $R^{(j_1)}$ ,  $R^{(j_2)}$  with dimensions  $2j_1 + 1$ ,  $2j_2 + 1$  can be reduced to a direct sum of (the complex conjugate of) representations  $R^{(j_3)*}$  [1]:

$$R_{m_1 m'_1}^{(j_1)}(x) R_{m_2 m'_2}^{(j_2)}(x) = \sum_{j_3 m_3 m'_3} (2j_3 + 1) \begin{pmatrix} j_1 & j_2 & j_3 \\ m_1 & m_2 & m_3 \end{pmatrix} R_{m_3 m'_3}^{(j_3)*}(x) \begin{pmatrix} j_1 & j_2 & j_3 \\ m'_1 & m'_2 & m'_3 \end{pmatrix}, \quad (1)$$

with  $x \in SU(2)$ . The  $j_1, j_2, j_3$  are half integers and  $m_1 = -j_1, -j_1 + 1, \dots, j_1$ , etc. Obviously Eq. (1) depends on the explicit choice of the representations  $R^{(j)}$ . As usual we use Euler parameters to parameterize  $SU(2)$  and choose representations that are diagonal in the first and the third rotation:

$$R_{mm'}^j(\alpha, \beta, \gamma) = \exp(im\alpha + im'\gamma) d_{mm'}^j(\beta), \quad (2)$$

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where the  $d_{mm'}^j$  are related to the Jacobi polynomials by

$$d_{mm'}^j(\beta) = \sqrt{\frac{(j+m)!(j-m)!}{(j+m')!(j-m')!}} \left(\cos \frac{\beta}{2}\right)^{m+m'} \left(\sin \frac{\beta}{2}\right)^{m-m'} P_{j-m}^{(m-m', m+m')}(\cos(\beta)) . \quad (3)$$

For the 3- $j$ -symbol a generating function is well known [2]:

$$\sum_{\alpha\beta\gamma} \begin{pmatrix} a & b & c \\ \alpha & \beta & \gamma \end{pmatrix} \sqrt{\frac{(a+b+c+1)!(a+b-c)!(b+c-a)!(c+a-b)!}{(a+\alpha)!(a-\alpha)!(b+\beta)!(b-\beta)!(c+\gamma)!(c-\gamma)!}} . \quad (4)$$

$$\cdot A^{a+\alpha} \bar{A}^{a-\alpha} B^{b+\beta} \bar{B}^{b-\beta} C^{c+\gamma} \bar{C}^{c-\gamma} = (A\bar{B} - B\bar{A})^{a+b-c} (B\bar{C} - C\bar{B})^{b+c-a} (C\bar{A} - A\bar{C})^{c+a-b} ,$$

where  $A, \bar{A}, B, \bar{B}, C, \bar{C}$  are all independent parameters.

If we divide both sides of this equation by  $(a+b-c)!(b+c-a)!(c+a-b)!$ , sum over  $a, b$ , and  $c$ , and introduce the shorthand

$$\Delta(a, b, c) = \left( \frac{(a+b-c)!(b+c-a)!(c+a-b)!}{(a+b+c+1)!} \right)^{\frac{1}{2}} \quad (5)$$

we find

$$\sum_{\substack{abc \\ \alpha\beta\gamma}} \begin{pmatrix} a & b & c \\ \alpha & \beta & \gamma \end{pmatrix} \frac{A^{a+\alpha} \bar{A}^{a-\alpha} B^{b+\beta} \bar{B}^{b-\beta} C^{c+\gamma} \bar{C}^{c-\gamma}}{\Delta(a, b, c) \sqrt{(a+\alpha)!(a-\alpha)!(b+\beta)!(b-\beta)!(c+\gamma)!(c-\gamma)!}}$$

$$= \sum_{a+b+c=0}^{\infty} \frac{(A\bar{B} - B\bar{A} + B\bar{C} - C\bar{B} + C\bar{A} - A\bar{C})^{a+b+c}}{(a+b+c)!} = \exp \left( \begin{vmatrix} 1 & A & \bar{A} \\ 1 & B & \bar{B} \\ 1 & C & \bar{C} \end{vmatrix} \right) . \quad (6)$$

Note that we have six expansion parameters  $A, \bar{A}, B, \bar{B}, C, \bar{C}$  for six variables  $a, \alpha, b, \beta, c, \gamma$ . We do not lose access to the 3- $j$ -symbol by the extra sums over  $a, b, c$ ; the 3- $j$ -symbol can be extracted by expanding the right hand side of Eq. (6). This equation reveals the full symmetry structure of the 3- $j$ -symbol including the Regge symmetries [2]  $a \pm \alpha \rightarrow b + c - a$ ,  $b \pm \beta \rightarrow c + a - b$ ,  $c \pm \gamma \rightarrow a + b - c$ , corresponding to  $A \rightarrow A\sqrt{\bar{B}\bar{C}/\bar{A}}$ ,  $\bar{A} \rightarrow \sqrt{\bar{B}\bar{C}/\bar{A}}$  plus cyclic permutations or  $A \rightarrow \sqrt{\bar{B}\bar{C}/\bar{A}}$ ,  $\bar{A} \rightarrow \bar{A}\sqrt{\bar{B}\bar{C}/\bar{A}}$  (+ cycl.), respectively.

Of course, Eq. (6) is not a generating function of the 3- $j$ -symbol itself, but a generating function of the 3- $j$ -symbol with a suitable normalization. This normalization eliminates the square roots in the 3- $j$ -symbol and, obviously, without the normalization calculating a generating function would not be realistic.

If we substitute  $\bar{A} \rightarrow t_1 \bar{A}$ ,  $\bar{B} \rightarrow t_2 \bar{B}$ ,  $\bar{C} \rightarrow t_3 \bar{C}$  in Eq. (6), multiply both sides by  $\exp(-t_1 - t_2 - t_3)$ , and integrate over  $t_1, t_2, t_3$  from 0 to  $\infty$  we obtain another, equivalent generating function for the 3- $j$ -symbol:

$$\sum_{\substack{abc \\ \alpha\beta\gamma}} \begin{pmatrix} a & b & c \\ \alpha & \beta & \gamma \end{pmatrix} \sqrt{\frac{(a-\alpha)!(b-\beta)!(c-\gamma)!}{(a+\alpha)!(b+\beta)!(c+\gamma)!}} \frac{A^{a+\alpha} \bar{A}^{a-\alpha} B^{b+\beta} \bar{B}^{b-\beta} C^{c+\gamma} \bar{C}^{c-\gamma}}{\Delta(a, b, c)}$$

$$= \frac{1}{1 + (B - C)\bar{A}} \frac{1}{1 + (C - A)\bar{B}} \frac{1}{1 + (A - B)\bar{C}} . \quad (7)$$

In the same spirit we also find generating functions for the 6- $j$ -symbol and the 9- $j$ -symbol:

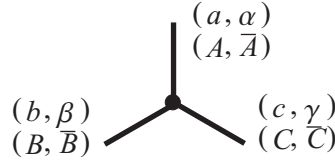
$$\begin{aligned}
& \sum_{\substack{abc \\ def}} \left\{ \begin{array}{ccc} a & b & c \\ d & e & f \end{array} \right\} \frac{A^{2a} B^{2b} C^{2c} D^{2d} E^{2e} F^{2f}}{\Delta(a, b, c) \Delta(a, e, f) \Delta(c, d, e) \Delta(b, d, f)} \\
&= (1 + ABF + ACE + BCD + DEF + ABDE + ACDF + BCEF)^{-2} , \tag{8} \\
& \sum_{\substack{abc \\ def \\ ghk}} \left\{ \begin{array}{ccc} a & b & c \\ d & e & f \\ g & h & k \end{array} \right\} \frac{A^{2a} B^{2b} C^{2c} D^{2d} E^{2e} F^{2f} G^{2g} H^{2h} K^{2k}}{\Delta(a, b, c) \Delta(d, e, f) \Delta(g, h, k) \Delta(a, d, g) \Delta(b, e, h) \Delta(c, f, k)} \\
&= (1 - ABDE - ABGH - ACDF - ACGK - BCEF - BCHK - DEGH - DFGK - EFHK + \\
&+ ABEFGK + ACDEHK + BCDFGH - ABDFHK - ACEFGH - BCDEGK)^{-2} . \tag{9}
\end{aligned}$$

In the next section we will find a simple geometric interpretation of these formulae. In Sec. 3 we will prove a theorem that incorporates Eqs. (7), (8), (9). We will find that by simple geometrical means it is possible to find a generating function for any multi- $j$ -symbol.

## 2 Graphical Notation

### 2.1 Graphical Notation for Multi- $j$ -Symbols

A 3- $j$ -symbol is denoted by a three-valent vertex where the external lines carry the indices  $(a, \alpha)$ ,  $(b, \beta)$ ,  $(c, \gamma)$  of the 3- $j$ -symbol (or  $(A, \bar{A})$ ,  $(B, \bar{B})$ ,  $(C, \bar{C})$  for the generating function):



Note that the 3- $j$ -symbol may pick up a minus sign under odd permutations of the columns. To keep track of those signs we implement the rule that the legs are labeled in a counter-clockwise orientation. Permuting to legs of the 3- $j$ -symbol amounts to a factor  $(-1)^{a+b+c}$ , which means on the level of generating functions  $A, B, C \rightarrow -A, -B, -C$ , or, since  $\alpha + \beta + \gamma = 0$ , equivalently  $\bar{A}, \bar{B}, \bar{C} \rightarrow -\bar{A}, -\bar{B}, -\bar{C}$ .

Two external lines  $(a_1, \alpha_1)$  and  $(a_2, \alpha_2)$  may be glued together with the group invariant 'metric'

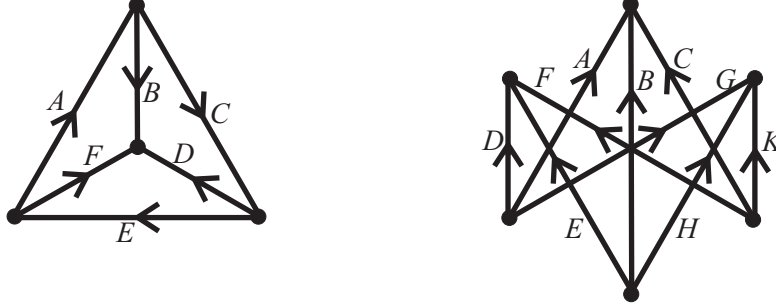
$$\begin{pmatrix} a_1 \\ \alpha_1, \alpha_2 \end{pmatrix} \delta_{a_{12}, a_1} \delta_{a_{12}, a_2} = (-1)^{a_1 + \alpha_1} \delta_{\alpha_1, -\alpha_2} \delta_{a_{12}, a_1} \delta_{a_{12}, a_2} \tag{10}$$



and sums over  $a_1$ ,  $a_2$ ,  $\alpha_1$ , and  $\alpha_2$ . The glued line has the angular momentum  $a_{12}$  and no magnetic quantum number. The metric is not symmetric under exchange of  $(a_1, \alpha_1)$  and  $(a_2, \alpha_2)$ . It thus has an orientation which is denoted by the arrow. Changing the orientation of the arrow amounts in a factor  $(-1)^{2a_{12}}$  or, on the level of the generating function  $(\sum_{a_{12}} f(a_{12}) A_{12}^{2a_{12}})$ , to  $A_{12} \rightarrow -A_{12}$ .

With these rules every multi- $j$ -symbol translates into a three-valent graph which we will call  $\Gamma$ . Every line in  $\Gamma$  has an angular momentum quantum number (a  $j$  variable, here labeled  $a, b, c, \dots$ , or  $a_1, a_2, a_3, \dots$ ). In addition, every external line has a magnetic quantum number (an  $m$  variable, here labeled  $\alpha, \beta, \gamma, \dots$ , or  $\alpha_1, \alpha_2, \alpha_3, \dots$ ). If there are no external lines the multi- $j$ -symbol is closed, it has no magnetic quantum numbers. Closed multi- $j$ -symbols have a group theoretical meaning independent of the specific representations chosen (Eq. (2)). However, we do not restrict ourselves to this case.

The standard 6- $j$ -, and 9- $j$ -symbols have the graphs:



These are the most interesting however by no means the only symbols with six or nine angular momenta.

## 2.2 Graphical Notation for the Generating Functions

Now we translate the quantum numbers  $a, \alpha$  into expansion coefficients  $A, \bar{A}$  according to the following rules:

For internal lines we construct the generating function by multiplying with  $A^{2a}$  and summing over  $2a = 0, 1, 2, \dots$ . This means we change variables from small letters to capitals.

For external lines we multiply with  $A^{a+\alpha}, \bar{A}^{a-\alpha}$  and sum over  $a + \alpha = 0, 1, 2, \dots$  and  $a - \alpha = 0, 1, 2, \dots$  (which is equivalent to  $a = 0, \frac{1}{2}, 1, \dots, \alpha = -a, -a+1, \dots, a$ ). We thus transform from  $(a, \alpha)$  to  $(A, \bar{A})$ .

Graphically the generating function of a multi- $j$ -symbol is represented by the graph of the multi- $j$ -symbol with the internal lines labeled  $A, B, C, \dots$  and the external lines labeled  $(A, \bar{A}), (B, \bar{B}), (C, \bar{C}), \dots$ .

To derive the generating functions of the multi- $j$ -symbol we have to introduce the following definition.

### Definition 2.1.

A curve  $\omega$  running through a graph  $\Gamma$  (following the lines of the graph) leads to the product  $P(\omega)$  of the variables of all lines that the curve passes. This curve may start and end in external lines. In this case the curve has an orientation indicated by an arrow. The external line where the curve starts from is represented by the unbarred variable, whereas the terminal (external) line enters the product by its bared variable. If the curve is closed it has no external lines and no bared variables occur. Curves that start from or end in internal lines are not regarded.

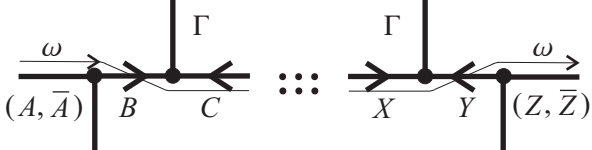
Moreover  $P(\omega)$  is endowed with a sign: It picks up a minus sign

1. for every time it passes a line against the orientation of its arrow,
2. for every time the direct way through a vertex (without crossing the third leg of the vertex) is a clockwise rotation, and

3. it has an over all minus sign.

For sets  $\Omega$  of curves  $\omega$  we define

$$P(\Omega) = \prod_{\omega \in \Omega} P(\omega) , \quad \text{if } \Omega \neq \emptyset \quad \text{and} \quad P(\emptyset) = 1 . \quad (11)$$



$$P(\omega) = (-1)^A (-1) B (+1) (-C) \cdots X (+1) (-Y) (-1) \bar{Z}$$

Note that for closed loops the direction we run through the loop is irrelevant: If we change the direction we pick up a phase  $(-1)^{\# \text{lines}} (-1)^{\# \text{vertices}} = 1$ . The first factor comes from running through the lines in opposite direction and the second factor stems from the fact that we reverse the orientation we run through the vertices when we reverse the direction of the loop. Rule 3. in the above definition means for sets  $\Omega$  of curves that  $P(\Omega)$  gets an over all sign  $(-1)^{\# \text{ connected components}}$ .

Now, we can give graphical notations for the generating function of the last section:

$$\sum_{\substack{abc \\ \alpha\beta\gamma}} \binom{a \ b \ c}{\alpha \ \beta \ \gamma} \frac{A^{a+\alpha} \bar{A}^{a-\alpha} B^{b+\beta} \bar{B}^{b-\beta} C^{c+\gamma} \bar{C}^{c-\gamma}}{\Delta(a, b, c) \sqrt{(a+\alpha)! (a-\alpha)! (b+\beta)! (b-\beta)! (c+\gamma)! (c-\gamma)!}}$$

$$= \exp \left( - \left( \begin{array}{c} \uparrow \uparrow (A, \bar{A}) \\ \swarrow \searrow \\ (B, \bar{B}) \quad (C, \bar{C}) \end{array} \right) \right) . \quad (12)$$

$$\sum_{\substack{abc \\ \alpha\beta\gamma}} \binom{a \ b \ c}{\alpha \ \beta \ \gamma} \sqrt{\frac{(a-\alpha)! (b-\beta)! (c-\gamma)!}{(a+\alpha)! (b+\beta)! (c+\gamma)!}} \frac{A^{a+\alpha} \bar{A}^{a-\alpha} B^{b+\beta} \bar{B}^{b-\beta} C^{c+\gamma} \bar{C}^{c-\gamma}}{\Delta(a, b, c)}$$

$$= \frac{1}{1 + \begin{array}{c} \uparrow \uparrow \bar{A} \\ \swarrow \searrow \\ B \quad C \end{array}} \frac{1}{1 + \begin{array}{c} \uparrow \uparrow A \\ \swarrow \searrow \\ \bar{B} \quad C \end{array}} \frac{1}{1 + \begin{array}{c} \uparrow \uparrow A \\ \swarrow \searrow \\ B \quad \bar{C} \end{array}} . \quad (13)$$

$$\sum_{\substack{abc \\ def}} \left\{ \begin{array}{ccc} a & b & c \\ d & e & f \end{array} \right\} \frac{A^{2a} B^{2b} C^{2c} D^{2d} E^{2e} F^{2f}}{\prod_{\substack{\text{vertices} \\ v_1, v_2, v_3 \in a, b, \dots, f}} \Delta(v_1, v_2, v_3)} = \left( 1 + \sum_{\substack{\text{non-overlapping} \\ \text{closed loops } \omega}} P(\omega) \right)^{-2} \quad (14)$$

$$\sum_{\substack{abc \\ def \\ ghk}} \left\{ \begin{array}{ccc} a & b & c \\ d & e & f \\ g & h & k \end{array} \right\} \frac{A^{2a} B^{2b} C^{2c} D^{2d} E^{2e} F^{2f} G^{2g} H^{2h} K^{2k}}{\prod_{\substack{\text{vertices} \\ v_1, v_2, v_3 \in a, b, \dots, f}} \Delta(v_1, v_2, v_3)} = \left( 1 + \sum_{\substack{\text{non-overlapping} \\ \text{closed loops } \omega}} P(\omega) \right)^{-2} . \quad (15)$$

The generating function is normalized by the square roots  $\Delta(v_1, v_2, v_3)$  (Eq. (5)) for all vertices  $v_1, v_2, v_3$  in  $\Gamma$ . The sum over all non-overlapping closed loops means the sum over all closed loops

$\omega$  that pass each line at most once. (In the next section we will see that in general we have to deal with sets of loops which may have more than one connected component.) The polynomial  $P(\omega)$  is of order one in each variable.

### 3 The Theorem

We start from Eq. (13) which represents the building block for a general multi- $j$ -symbol.

A generating function  $F(A_1, \bar{A}_1, A_2, \bar{A}_2)$  with two external lines  $(A_1, \bar{A}_1)$  and  $(A_2, \bar{A}_2)$  is glued together to yield a generating function  $\tilde{F}(A_{12})$  with an internal line  $A_{12}$  (running from  $A_1$  to  $A_2$ ) by the following procedure:

$$\tilde{F}(A_{12}) = \oint_{\partial U} \frac{dc_1}{2\pi i c_1} \oint_{\partial U} \frac{dc_2}{2\pi i c_2} F(-A_{12}c_1, c_2^{-1}, A_{12}c_2, c_1^{-1}) . \quad (16)$$

The integrals on the right hand side are loop integrals around the unit circle  $\partial U$  of the complex plane. The integrals may be evaluated using the residue theorem assuming  $A_{12}$  (being an expansion parameter) is small.

It is readily checked that Eqs. (10) and (16) are equivalent:

$$\begin{aligned} & \oint_{\partial U} \frac{dc_1}{2\pi i c_1} \oint_{\partial U} \frac{dc_2}{2\pi i c_2} F(-A_{12}c_1, c_2^{-1}, A_{12}c_2, c_1^{-1}) \\ &= \sum_{a_1 \alpha_1 a_2 \alpha_2} F_{a_1 \alpha_1 a_2 \alpha_2} \oint_{\partial U} \frac{dc_1}{2\pi i c_1} \oint_{\partial U} \frac{dc_2}{2\pi i c_2} (-A_{12}c_1)^{a_1 + \alpha_1} (c_2)^{-a_1 + \alpha_1} (A_{12}c_2)^{a_2 + \alpha_2} (c_1)^{-a_2 + \alpha_2} \\ &= \sum_{a_1 \alpha_1 a_2 \alpha_2} F_{a_1 \alpha_1 a_2 \alpha_2} (-1)^{a_1 + \alpha_1} A_{12}^{a_1 + \alpha_1 + a_2 + \alpha_2} \delta_{a_1 + \alpha_1, a_2 - \alpha_2} \delta_{a_1 - \alpha_1, a_2 + \alpha_2} \\ &= \sum_{a_1 \alpha_1 a_2 \alpha_2} F_{a_1 \alpha_1 a_2 \alpha_2} A_{12}^{2a_{12}} (-1)^{a_1 + \alpha_1} \delta_{\alpha_1, -\alpha_2} \delta_{a_{12}, a_1} \delta_{a_{12}, a_2} . \end{aligned}$$

Note that the identifications  $a_1 = a_2$ ,  $\alpha_1 = -\alpha_2$  cancel the square roots  $\sqrt{\frac{(a_1 + \alpha_1)!}{(a_1 - \alpha_1)!}} \sqrt{\frac{(a_2 + \alpha_2)!}{(a_2 - \alpha_2)!}}$ .

As an example let us glue two 3- $j$ -symbols to obtain a 5- $j$ -symbol with four external lines:

$$\begin{aligned} & \sum_{\substack{abcde \\ \alpha_1 \beta \gamma \alpha_2 \delta \varepsilon}} \begin{pmatrix} a & b & c \\ \alpha_1 & \beta & \gamma \end{pmatrix} \begin{pmatrix} a & d & e \\ \alpha_2 & \delta & \varepsilon \end{pmatrix} \begin{pmatrix} a \\ \alpha_1, \alpha_2 \end{pmatrix} \sqrt{\frac{(b - \beta)!(c - \gamma)!}{(b + \beta)!(c + \gamma)!}} \sqrt{\frac{(d - \delta)!(e - \varepsilon)!}{(d + \delta)!(e + \varepsilon)!}} \\ & \cdot \frac{A^{2a} B^{b+\beta} \bar{B}^{b-\beta} C^{c+\gamma} \bar{C}^{c-\gamma} D^{d+\delta} \bar{D}^{d-\delta} E^{e+\varepsilon} \bar{E}^{e-\varepsilon}}{\Delta(a, b, c) \Delta(a, d, e)} \\ &= \oint_{\partial U} \frac{dc_1}{2\pi i c_1} \oint_{\partial U} \frac{dc_2}{2\pi i c_2} \frac{1}{1 + (B - C) c_2^{-1}} \frac{1}{1 + (C + A c_1) \bar{B}} \\ & \cdot \frac{1}{1 + (-A c_1 - B) \bar{C}} \frac{1}{1 + (D - E) c_1^{-1}} \frac{1}{1 + (E - A c_2) \bar{D}} \frac{1}{1 + (A c_2 - D) \bar{E}} \\ &= \frac{1}{1 + (C - A(D - E)) \bar{B}} \frac{1}{1 + (A(D - E) - B) \bar{C}} \\ & \cdot \frac{1}{1 + (E + A(B - C)) \bar{D}} \frac{1}{1 + (-A(B - C) - D) \bar{E}} \end{aligned}$$

$$= \frac{1}{1 + \text{diagram}} + \frac{1}{1 + \text{diagram}} + \frac{1}{1 + \text{diagram}} + \frac{1}{1 + \text{diagram}},$$

The diagrams are three-valent graphs with external lines labeled A, B, C, D, E, and internal lines labeled with overbars. The first diagram has external lines A, B, C, D, E with internal lines B-bar, C-bar, D-bar. The second has A, B, C, D, E with internal lines B-bar, C-bar, D-bar. The third has A, B, C, D, E with internal lines B-bar, C-bar, D-bar. The fourth has A, B, C, D, E-bar with internal lines B-bar, C-bar, D-bar.

where we used the residue theorem and evaluated the residues inside the unit circle.

Before we formulate the general case we need some more notation.

**Notation 3.1.**

Let  $\Gamma$  be the graph of the multi- $j$ -symbol  $S(\Gamma)$  with external lines  $a_j$ ,  $j = 1, 2, \dots, J$  and internal lines  $a_i$ ,  $i = J + 1, J + 2, \dots, J + I$ . If  $J = 0$  then  $\Gamma$  is closed.

Let  $L = I + J$  be the number of lines and  $V$  the number of vertices in  $\Gamma$ .

We consider curves  $\omega$  in  $\Gamma$  as defined in Def. 2.1. A set  $\Omega$  of curves in  $\Gamma$  is said to run from  $A_i$  to  $A_j$ ,  $i, j = 1, \dots, J$  if one curve in  $\Omega$  is open and runs from  $A_i$  to  $A_j$  and all other curves in  $\Omega$  are closed loops.

The degree  $\deg_{A_i}(\omega)$  of a line  $A_i$  in a curve  $\omega$  is the number of times  $\omega$  passes through this line. The degree  $\deg_{A_i}(\Omega)$  of a line  $A_i$  in a set  $\Omega$  of curves is the sum  $\sum_{\omega \in \Omega} \deg_{A_i}(\omega)$ . A (set of) curve(s)  $\omega$  ( $\Omega$ ) is non-overlapping if  $\deg_{A_i}(\omega) \leq 1 \ \forall A_i \in \Gamma$  ( $\deg_{A_i}(\Omega) \leq 1 \ \forall A_i \in \Gamma$ ).

Let  $\Omega_{ij}$ ,  $i, j = 1, \dots, J$  be the set of all non-overlapping sets of curves in  $\Gamma$  running from  $A_i$  to  $A_j$ .

Let  $\Omega_0$  be the set of all not overlapping sets of closed loops in  $\Gamma$  (including the empty set  $\emptyset \in \Omega_0$ ).

Let  $P(\omega)$  ( $P(\Omega)$ ), as defined in Def. 2.1, be the sign endowed product of all lines  $\omega$  ( $\Omega$ ) runs through. We have  $\deg_{A_i}(P(\Omega)) = \deg_{A_i}(\Omega)$ .

We can draw some elementary conclusions from the fact that  $\Gamma$  has only three-valent vertices.

**Remark 3.2.**

1.  $J + 2I = 3V$ . Moreover  $J = V + 2 \iff \Gamma$  is a tree diagram.

2.

$$|\Omega_0| = 2^{I-V+1} = 2^{(V+2-J)/2}. \quad (17)$$

This is easily seen by induction. For  $J = V + 2$  we obviously have  $\Omega_0 = \{\emptyset\}$ ,  $|\Omega_0| = 1$ . If we reduce  $J$  by two via gluing of two external lines then  $\Omega_0$ , <sub>glued</sub> splits into two parts. One where the glued line has degree zero and one where it has degree one. The number of sets in the first part is obviously  $\Omega_0$ , <sub>unglued</sub>. The second part contains at least one loop. However there is a one to one correspondence between  $\Omega_0$ , <sub>unglued</sub> and sets of curves running through the glued line: If we have two sets  $\Omega_1$ ,  $\Omega_2$  of non-overlapping curves with degree one at the glued vertex we can take the 'difference' of these sets by reducing the degrees of the lines in  $\Omega_1 \cup \Omega_2 \mod 2$ . This defines a set of loops in  $\Omega_0$ , <sub>unglued</sub>. Thus  $|\Omega_0$ , <sub>glued</sub> $| = 2 \cdot |\Omega_0$ , <sub>unglued</sub> $|$ .

3.

$$|\Omega_{ij}| = |\Omega_0|, \text{ if } J \geq 2. \quad (18)$$

In deed, if we glue  $A_i$  and  $A_j$  we see that  $|\Omega_{ij}|$  equals the number of sets in the glued graph with degree one at the glued line. This is  $|\Omega_0|$ , as explained above.

**Theorem 3.3.** With the above notation and  $\Delta(a, b, c)$  as in Eq. (5) we obtain a generating function for  $S(\Gamma)$  by

$$\begin{aligned} & \sum_{\substack{a_j \alpha_j, j \leq J \\ a_i, i > J}} S(\Gamma, a_j, \alpha_j, a_i) \frac{\prod_{j \leq J} A_j^{a_j + \alpha_j} \bar{A}_j^{a_j - \alpha_j} \prod_{i > J} A_i^{2a_i}}{\prod_{\substack{\text{vertices } a_k, a_\ell, a_m \\ k, \ell, m=1, \dots, L}} \Delta(a_k, a_\ell, a_m)} \prod_{j \leq J} \sqrt{\frac{(a_j - \alpha_j)!}{(a_j + \alpha_j)!}} \\ &= \left( \sum_{\omega \in \Omega_0} P(\omega) \right)^{|J|-2} \prod_{j \leq J} \left( \sum_{\substack{\omega \in \Omega_0 \cup \Omega_{ij} \\ i \neq j}} P(\omega) \right)^{-1}. \end{aligned} \quad (19)$$

**Proof.** We will prove the theorem in two steps.

First, we show the theorem is valid for tree graphs with  $\Omega_0 = \{\emptyset\}$ . This can easily be done by induction over the number of vertices in  $\Gamma$ . Obviously Eq. (19) is valid for the 3- $j$ -symbol, Eq. (7). The gluing of further 3- $j$ -symbols follows closely the example of the 5- $j$ -symbol in the previous section. Evaluating each loop integral amounts to a substitution and the orientation of the glued line is taken care of by the minus sign in the gluing prescription.

Second, we have to show that Eq. (19) remains valid under gluing of any two external lines in  $\Gamma$ . The general result follows by induction over the number of times gluing is necessary. So, assume Eq. (19) is valid and we want to glue (without restriction)  $A_1$  and  $A_2$ . Let  $P_0(\Omega)$  be the  $A_1, \bar{A}_1, A_2, \bar{A}_2$ -independent part of  $P(\Omega)$ . The first factor on the right hand side of Eq. (19) is independent of  $A_1, \bar{A}_1, A_2, \bar{A}_2$ . The second factor is

$$\begin{aligned} & \left( \sum_{\Omega \in \Omega_0} P(\Omega) + \sum_{\Omega \in \Omega_{21}} P_0(\Omega) A_2 \bar{A}_1 + \sum_{\Omega \in \Omega_{i1}, i \neq 1, 2} P_0(\Omega) \bar{A}_1 \right)^{-1} \\ & \cdot \left( \sum_{\Omega \in \Omega_0} P(\Omega) + \sum_{\Omega \in \Omega_{12}} P_0(\Omega) A_1 \bar{A}_2 + \sum_{\Omega \in \Omega_{i2}, i \neq 1, 2} P_0(\Omega) \bar{A}_2 \right)^{-1} \\ & \cdot \prod_{2 < j \leq J} \left( \sum_{\Omega \in \Omega_0} P(\Omega) + \sum_{\Omega \in \Omega_{1j}} P_0(\Omega) A_1 + \sum_{\Omega \in \Omega_{2j}} P_0(\Omega) A_2 + \sum_{\substack{\Omega \in \Omega_{ij} \\ i \neq 1, 2, j}} P(\Omega) \right)^{-1}. \end{aligned}$$

According to the gluing prescription, Eq. (16), we now substitute  $A_1 \rightarrow -A_{12}c_1$ ,  $\bar{A}_1 \rightarrow c_2^{-1}$ ,  $A_2 \rightarrow A_{12}c_2$ ,  $\bar{A}_2 \rightarrow c_1^{-1}$  and multiply by  $\frac{1}{2\pi i c_1} \frac{1}{2\pi i c_2}$ . The loop integrals over  $c_1$  and  $c_2$  amount to substituting the first and the second factor into the product over  $j > 2$  (we assume all  $A_i, A_j$  are small and evaluate the residues inside the unit circle). To be precise we obtain

$$\begin{aligned} & \left( \sum_{\Omega \in \Omega_0} P(\Omega) + \sum_{\Omega \in \Omega_{21}} A_{12} P_0(\Omega) \right)^{-1} \left( \sum_{\Omega \in \Omega_0} P(\Omega) - \sum_{\Omega \in \Omega_{12}} A_{12} P_0(\Omega) \right)^{-1} \cdot \prod_{2 < j \leq J} \\ & \left( \sum_{\Omega \in \Omega_0} P(\Omega) + \sum_{\substack{\Omega \in \Omega_{ij} \\ i \neq 1, 2, j}} P(\Omega) + \frac{\sum_{\Omega \in \Omega_{1j}} A_{12} P_0(\Omega) \sum_{\substack{\Omega \in \Omega_{i2} \\ i \neq 1, 2}} P_0(\Omega)}{\sum_{\Omega \in \Omega_0} P(\Omega) - \sum_{\Omega \in \Omega_{12}} A_{12} P_0(\Omega)} - \frac{\sum_{\Omega \in \Omega_{2j}} A_{12} P_0(\Omega) \sum_{\substack{\Omega \in \Omega_{i1} \\ i \neq 1, 2}} P_0(\Omega)}{\sum_{\Omega \in \Omega_0} P(\Omega) + \sum_{\Omega \in \Omega_{21}} A_{12} P_0(\Omega)} \right)^{-1}. \end{aligned}$$



If we reverse the direction of the open curve in  $\Omega$  we obtain  $\sum_{\Omega \in \Omega_{21}} P_0(\Omega) = -\sum_{\Omega \in \Omega_{12}} P_0(\Omega)$ . We denote the glued graph with  $\Gamma'$  and the set of all non-overlapping closed loops in  $\Gamma'$  with  $\Omega'_0$ . Analogously we define  $\Omega'_{ij}$  as the set of non-overlapping sets of curves in  $\Gamma'$  running from  $A_i$  to  $A_j$ . Thus

$$\sum_{\Omega \in \Omega_0} P(\Omega) + \sum_{\Omega \in \Omega_{21}} A_{12} P_0(\Omega) = \sum_{A_{12} \notin \Omega \in \Omega'_0} P(\Omega) + \sum_{A_{12} \in \Omega \in \Omega'_0} P(\Omega) = \sum_{\Omega \in \Omega'_0} P(\Omega) .$$

This simplifies the above result to

$$\left( \sum_{\Omega \in \Omega'_0} P(\Omega) \right)^{J-4} \prod_{2 < j \leq J} \left( \left( \sum_{\Omega \in \Omega_0} P(\Omega) \right) \left( \sum_{\Omega \in \Omega'_0} P(\Omega) + \sum_{\substack{\Omega \in \Omega_{ij} \\ i \neq 1,2,j}} P(\Omega) \right) \right. \\ \left. + \sum_{A_{12} \in \Omega \in \Omega'_0} P(\Omega) \sum_{\substack{\Omega \in \Omega_{ij} \\ i \neq 1,2,j}} P(\Omega) + \sum_{\Omega \in \Omega_{1j}} A_{12} P_0(\Omega) \sum_{\substack{\Omega \in \Omega_{i2} \\ i \neq 1,2}} P_0(\Omega) - \sum_{\Omega \in \Omega_{2j}} A_{12} P_0(\Omega) \sum_{\substack{\Omega \in \Omega_{i1} \\ i \neq 1,2}} P_0(\Omega) \right)^{-1} .$$

We need the following notation:

Let  $\Omega'_{i12}$  be the set of non-overlapping curves in  $\Gamma'$  running from  $A_i$  to  $A_{12}$  parallel to the orientation of  $A_{12}$ .

Let  $\Omega'_{i21}$  be the set of non-overlapping curves in  $\Gamma'$  running from  $A_i$  to  $A_{12}$  anti-parallel to the orientation of  $A_{12}$ .

Let  $\Omega'_{12j}$  be the set of non-overlapping curves in  $\Gamma'$  running from the  $A_2$  vertex of  $A_{12}$  to  $A_j$  without passing through  $A_{12}$ .

Let  $\Omega'_{21j}$  be the set of non-overlapping curves in  $\Gamma'$  running from the  $A_1$  vertex of  $A_{12}$  to  $A_j$  without passing through  $A_{12}$ .

With these notations  $\Omega'_{i12} \cup \Omega'_{12j}$  contains a curve running from  $A_i$  to  $A_j$  passing  $A_{12}$  parallel to the orientation of  $A_{12}$ , and  $\Omega'_{i21} \cup \Omega'_{21j}$  contains a curve running from  $A_i$  to  $A_j$  passing  $A_{12}$  anti-parallel to the orientation of  $A_{12}$ . Now the last three terms can be written as

$$\sum_{\substack{A_{12} \in \Omega_1 \in \Omega'_0 \\ A_{12} \notin \Omega_2 \in \Omega'_{ij}, i \neq 1,2,j}} P(\Omega_1 \cup \Omega_2) + \sum_{\substack{\Omega_1 \in \Omega'_{i21}, i \neq 1,2 \\ \Omega_2 \in \Omega'_{21j}}} P_0(\Omega_1 \cup \Omega_2) + \sum_{\substack{\Omega_1 \in \Omega'_{i12}, i \neq 1,2 \\ \Omega_2 \in \Omega'_{12j}}} P_0(\Omega_1 \cup \Omega_2) = X .$$

To get the signs right in this expression we have to bear in mind that in the last two terms we gain a minus sign by connecting the open curve in  $\Omega_1$  and  $\Omega_2$  and in the middle term we get another minus sign by running against the orientation of  $A_{12}$ . Now we can simplify  $X$  as follows: Whenever the degree of  $\Omega_1 \cup \Omega_2$  is 2 at some line  $A_k$  we can swap the end points of the lines to get a crossing instead of two parallel lines (and vice versa). This gives a new set  $\overline{\Omega_1 \cup \Omega_2}$ . We find  $P(\Omega_1 \cup \Omega_2) = -P(\overline{\Omega_1 \cup \Omega_2})$ , where the minus sign stems from either the change of orientations if we started from two anti-parallel lines or from gaining or losing a connected component if we started from parallel lines. A non-trivial but purely geometrical calculation leads to the expression

$$X = \sum_{\Omega \in \Omega_0} P(\Omega) - \sum_{\substack{A_{12} \in \Omega \in \Omega'_{ij} \\ i \neq 1,2,j}} P(\Omega) .$$

Altogether we obtain

$$\left( \sum_{\Omega \in \Omega_0} P(\Omega) \right)^{-J+2} \left( \sum_{\Omega \in \Omega'_0} P(\Omega) \right)^{J-4} \prod_{2 < j \leq J} \left( \sum_{\Omega \in \Omega'_0} P(\Omega) + \sum_{\substack{A_{12} \notin \Omega \in \Omega'_{ij} \\ i \neq 1, 2, j}} P(\Omega) + \sum_{\substack{A_{12} \in \Omega \in \Omega'_{ij} \\ i \neq 1, 2, j}} P(\Omega) \right).$$

The first factor cancels the prefactor in Eq. (19). The sums combine to  $\sum_{\Omega \in \Omega'_0 \cup \Omega'_{ij}, i \neq 1, 2, j} P(\Omega)$ , which establishes the desired result.  $\square$

It is possible to rewrite Eq. (19) in the spirit of Eq. (6) which makes it slightly more symmetric.

**Corollary 3.4.**

$$\begin{aligned} & \sum_{\substack{a_j \alpha_j, j \leq J \\ a_i, i > J}} S(\Gamma, a_j, \alpha_j, a_i) \frac{\prod_{j \leq J} A_j^{a_j + \alpha_j} \bar{A}_j^{a_j - \alpha_j} \prod_{i > J} A_i^{2a_i}}{\prod_{\substack{\text{vertices } a_k, a_\ell, a_m \\ k, l, m = 1, \dots, L}} \Delta(a_k, a_\ell, a_m)} \prod_{j \leq J} \frac{1}{\sqrt{(a_j - \alpha_j)! (a_j + \alpha_j)!}} \\ &= \left( \sum_{\omega \in \Omega_0} P(\omega) \right)^{-2} \exp \left( - \frac{\sum_{\omega \in \Omega_{ij}, i \neq j} P(\omega)}{\sum_{\omega \in \Omega_0} P(\omega)} \right). \end{aligned} \quad (20)$$

**Proof.** We proceed in the same way as we did when we derived Eq. (7) from Eq. (6). We substitute  $\bar{A}_i \rightarrow t_i \bar{A}_i \forall i > J$  in Eq. (20) and multiply both sides by  $\exp(-\sum_{i > J} t_i)$ . Integrating over the  $t_i$  from 0 to  $\infty$  yields Eq. (19). This proves the corollary since the transformation between both generating functions is obviously invertible.  $\square$

## 4 Results and Outlook

We found explicit geometric results for generating functions of multi- $j$ -symbols. This result provides closed expressions for the multi- $j$ -symbols themselves in terms of finite sums. The right hand side of Eq. (19) has the form

$$(1 + A)^{J-2} \prod_{j=1}^J (1 + A + B_j)^{-1}, \quad A = \sum_{\emptyset \neq \Omega \in \Omega_0} P(\Omega), \quad B = \sum_{\Omega \in \Omega_{ij}, i \neq j} P(\Omega). \quad (21)$$

This may be expanded as

$$\sum_{k_1=0}^{\infty} \cdots \sum_{k_J=0}^{\infty} (-B_j)^{k_j} (1 + A)^{-\sum_j k_j - 2} = \sum_{k_1=0}^{\infty} \cdots \sum_{k_J=0}^{\infty} (-B_j)^{k_j} \sum_{k_0=0}^{\infty} \binom{\sum_j k_j + k_0 + 1}{k_0} A^{k_0}. \quad (22)$$

Expanding  $A$  and  $B_j$  yields  $J(J-1)2^{(V+2-J)/2} + 2^{(V+2-J)/2} - 1$  sums (Eqs. (17), (18)). Comparing coefficients gives  $I + 2J$  relations. This means that a full expansion provides an expression for the multi- $j$ -symbols in terms of  $(J^2 - J + 1)2^{(V+2-J)/2} - 3(V+J)/2 - \delta_{J,0}$  finite sums. The Kronecker delta reflects the fact that, if  $J \neq 0$ , the sum over all magnetic quantum numbers is zero automatically; this identity does not reduce the number of independent sums. If we specify

to the 3- $j$ -symbol  $V = 1$ ,  $J = 3$ , or to the 6- $j$ -symbol  $V = 4$ ,  $J = 0$  we obtain single sums which are the well known results [1]

$$\begin{pmatrix} a & b & c \\ \alpha & \beta & \gamma \end{pmatrix} = \Delta(a, b, c) \sqrt{(a + \alpha)! (a - \alpha)! (b + \beta)! (b - \beta)! (c + \gamma)! (c - \gamma)!} \\ \sum_z \frac{(-1)^{z+a-b-\gamma} \delta_{\alpha+\beta+\gamma,0}}{z! (a + b - c - z)! (a - \alpha - z)! (b + \beta - z)! (c - b + \alpha + z)! (c - a - \beta + z)!} \quad (23)$$

$$\left\{ \begin{matrix} a & b & c \\ d & e & f \end{matrix} \right\} = \Delta(a, b, c) \Delta(a, e, f) \Delta(b, d, f) \Delta(c, d, e) \\ \sum_z \frac{(-1)^z (z + 1)!}{(z - a - b - c)! (z - a - e - f)! (z - b - d - f)! (z - c - d - e)!} \\ \cdot \frac{1}{(a + b + d + e - z)! (b + c + e + f - z)! (a + c + d + f - z)!} \cdot \quad (24)$$

The last equation has been derived from the first equation by Racah [3] with quite tedious calculations.

The analogous formula for the 9- $j$ -symbol contains already a six-fold finite sum and is therefore hardly of practical use. However the expansion of the generating function allows us to give a geometrical interpretation of the multi- $j$ -symbols themselves. Let us for simplicity stick to the closed case  $J = 0$ . The angular momenta  $a_1, \dots, a_L$  label the lines of the graph  $\Gamma$ . The multi- $j$ -symbol counts the number of different ways how non-overlapping sets of loops can be laid on top of each other so that the total degree of each line  $k$  is given by  $a_k$ . Each possible solution is weighted by one plus the number of layers needed. In addition it has a combinatorial factor of how many different ways the layers can be permuted.

The search for further applications of these results is not yet completed. It may help to evaluate multi loop Feynman diagrams since the result of the angular integrations is given by the square of a multi- $j$ -symbol (more than three-valent vertices are blown up to a chain of three-valent vertices to give the graph of a multi- $j$ -symbol).

Obvious generalizations are the application to higher rank Lie-groups or to quantum groups. These may be covered in future publications.

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## References

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